Emergence of collective behavior for excitable units in mean-field interaction

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Joint work with Christophe Poquet (Lyon 1).

Excitability in neuronal systems

The aim of the talk is to describe the collective behavior of identical excitable units under the influence of noise and interaction.
One is particularly interested in phenomena like

- Persistence of oscillatory behaviors (e.g. when individual oscillations persist for the whole macroscopic population)
- Emergence of macroscopic structured dynamics due to noise and interaction (when individual dynamics is not oscillatory)

Dynamics of an isolated unit

Consider $X_t \in \mathbb{R}^d$, $t \geqslant 0$, solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t = F(X_t),$$

where

$$\begin{aligned} \langle F(x) - F(y), x - y \rangle &\leqslant C\|x - y\|^2, \\ \max\left(\|F(x)\|, \|\nabla F(x)\|, \|\nabla^2 F(x)\|\right) &\leqslant C \exp\left(c\|x\|^2\right) \end{aligned}$$

Main example : FitzHugh-Nagumo dynamics

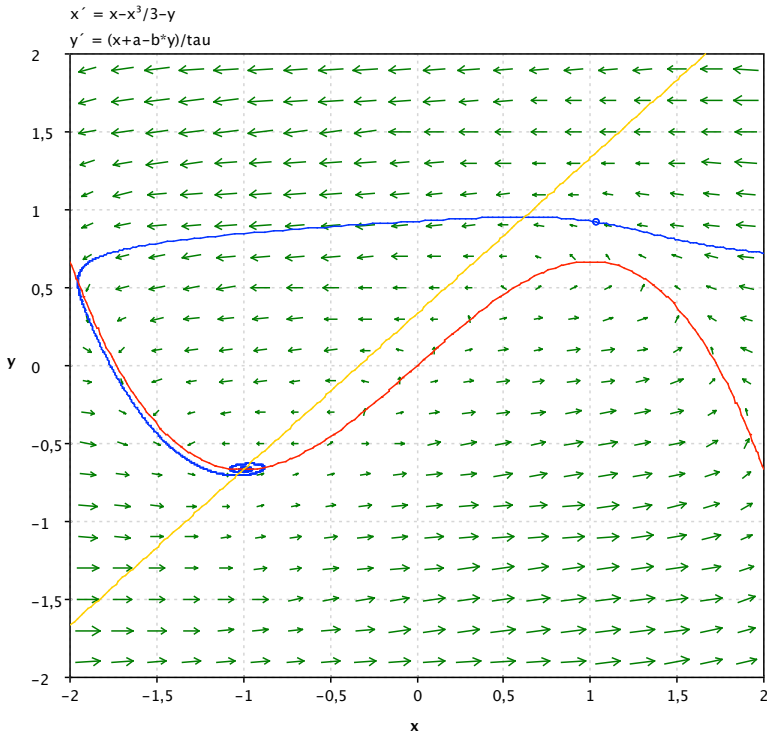
$$X = (v, w) \in \mathbb{R}^2, \ F(v, w) = \left(v - \frac{v^3}{3} - w, \frac{1}{\tau}(v + a - bw)\right).$$

But also Stuart-Landau-like oscillators, etc.

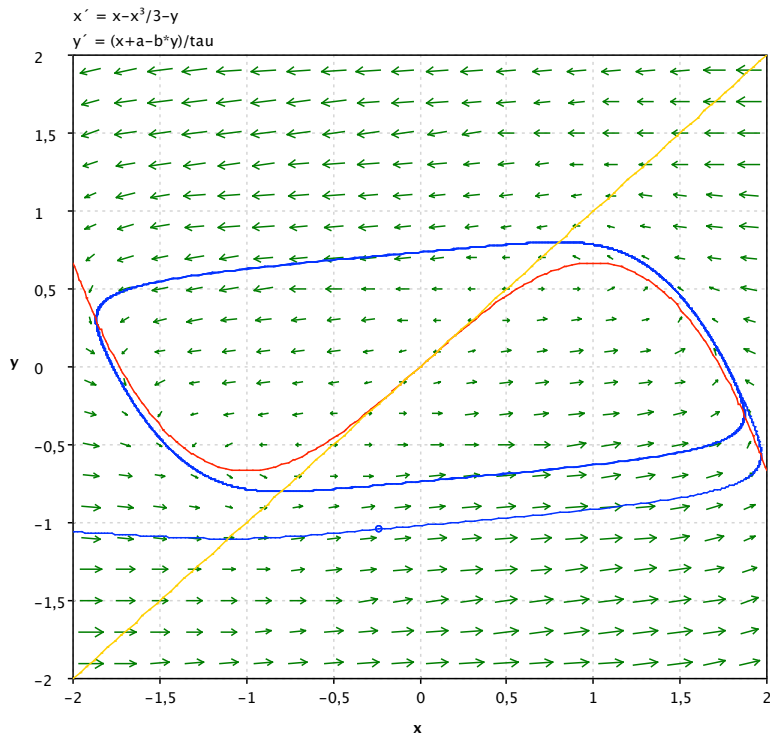
FitzHugh-Nagumo as an prototype for excitability

Excitable system [Lindner et al. 2004] : a system that

- without perturbation : rests in a stable state,
- with a sufficiently strong perturbation : the system leaves this resting state, resulting in a large excursion in the phase space.



(a) FHN model : resting (excitable) state



(b) FHN model : spiking state

Dynamics of connected units

Consider $N \gg 1$ such units $(X_{1,t}, \dots, X_{N,t})$, connected in a mean-field linear interaction and subject to individual noise :

$$\mathrm{d}X_{i,t} = \delta F(X_{i,t})\mathrm{d}t - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)\mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_{i,t}, i = 1, \dots, N$$

Usual propagation of chaos estimates shows that each $X_{i,t}$ is well-described as $N \to \infty$ (at least on a time-scale of order $T = O(\log(N))$) by the following nonlinear process

$$\mathrm{d}\bar{X}_t = \delta F(\bar{X}_t)\mathrm{d}t - K\left(\bar{X}_t - \mathbb{E}\left(\bar{X}_t\right)\right)\mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_t$$

Main question

Is the isolated dynamics (i.e. for $K = \sigma = 0$) significantly different from the connected dynamics ?

The law $\mu_t(\mathrm{d}x)$ of the nonlinear process $\bar{X}_t$ is a weak solution to the nonlinear Fokker-Planck equation

$$\partial_t \mu_t = \nabla\cdot\left(\sigma^2\nabla\mu_t\right)+\nabla\cdot\left(K\mu_t(x)\left(x-\int_{\mathbb{R}^d} z\mu_t(\mathrm{d}z)\right)-\delta\nabla\cdot\left(\mu_t(x)F(x)\right)\right).$$

Previous works

- [N. Berglund, B. Gentz, C.Kuehn] : with noise but no interaction
- [S. Mischler, C. Quiñinao, J. Touboul 2016] : long-time analysis for the FHN model in the kinetic case, in the $K \to 0$ regime.
- [C. Quiñinao, J. Touboul 2018] : long-time analysis for the kinetic FHN model when $\sigma \to 0$.

Two main differences here :

- We are not considering the kinetic case : noise and interaction and nontrivial on each coordinates
- We are looking at the regime of strong noise and interaction w.r.t. the individual dynamics, namely

$$K \text{ and } \sigma \text{ are fixed and } \delta \ll 1.$$

Taking $\delta \to 0$: a slow-fast analysis

$$\partial_t \mu_t = \nabla \cdot \left(\sigma^2 \nabla \mu_t\right) + \nabla \cdot \left(K \mu_t(x) \left(x - \int_{\mathbb{R}^d} z \mu_t(\mathrm{d}z) \right) - \delta \nabla \cdot (\mu_t(x) F(x)) \right).$$

The main point of the analysis is to decompose the process μ_t in terms of

- its mean value $m_t = \int_{\mathbb{R}^d} z \mu_t(\mathrm{d}z)$
- the centered process p_t defined by

$$\int_{\mathbb{R}^d} \varphi(x) p_t(\mathrm{d}x) := \int_{\mathbb{R}^d} \varphi(x - m_t) \mu_t(\mathrm{d}x).$$

Rewriting everything in terms of (p_t, m_t), we obtain

$$\left\{ \begin{array}{rcl} \partial_t p_t(x) & = & \nabla \cdot (\sigma^2 \nabla p_t(x)) + \nabla \cdot (p_t(x)(Kx + \dot{m}_t - \delta F(x + m_t))) \\ \dot{m}_t & = & \delta \int_{\mathbb{R}^d} F(x + m_t) p_t(\mathrm{d}x) \end{array} \right.$$

This is a slow-fast dynamics !

The case $\delta = 0$: a simple Ornstein-Uhlenbeck process

In the case $\delta = 0$, the mean-value $m_t = m_0$ is constant and the dynamics of the process p_t is simply given by

$$\partial_t p_t = \nabla \cdot (\sigma^2 \nabla p_t(x)) + \nabla \cdot (p_t(x) K x).$$

This is nothing else than the law of the Ornstein-Uhlenbeck process

$$\mathrm{d}\bar{X}_t = -K\bar{X}_t \mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_t$$

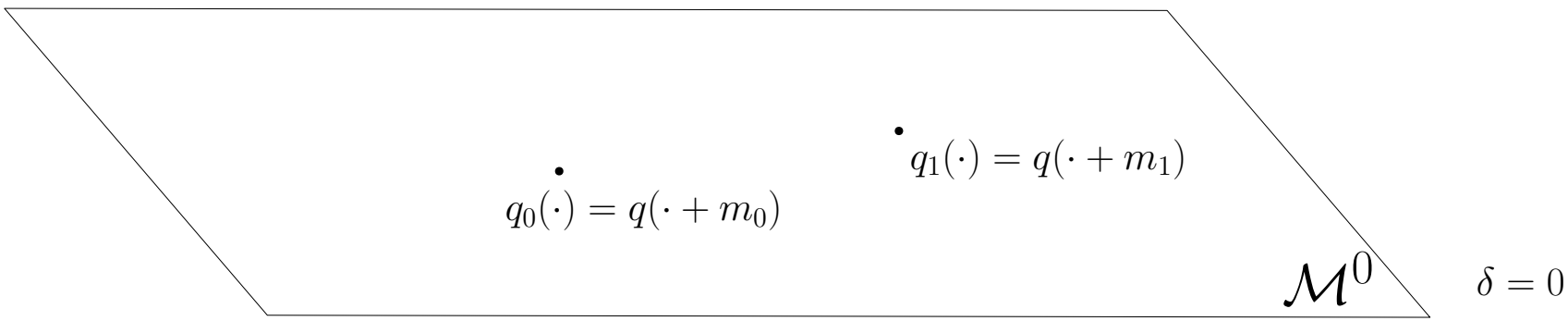
In this case, the dynamics is reversible and we have exponential convergence to a Gaussian invariant measure :

$$\|p_t - q\|_{L^2(w)} \leqslant e^{-k_0 t} \|p_0 - q\|_{L^2(w)} ,$$

for

$$q(x) = \frac{1}{((2\pi)^d \det(\sigma^2 K^{-1}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} x \cdot K\sigma^{-2} x\right), \ x \in \mathbb{R}^d .$$

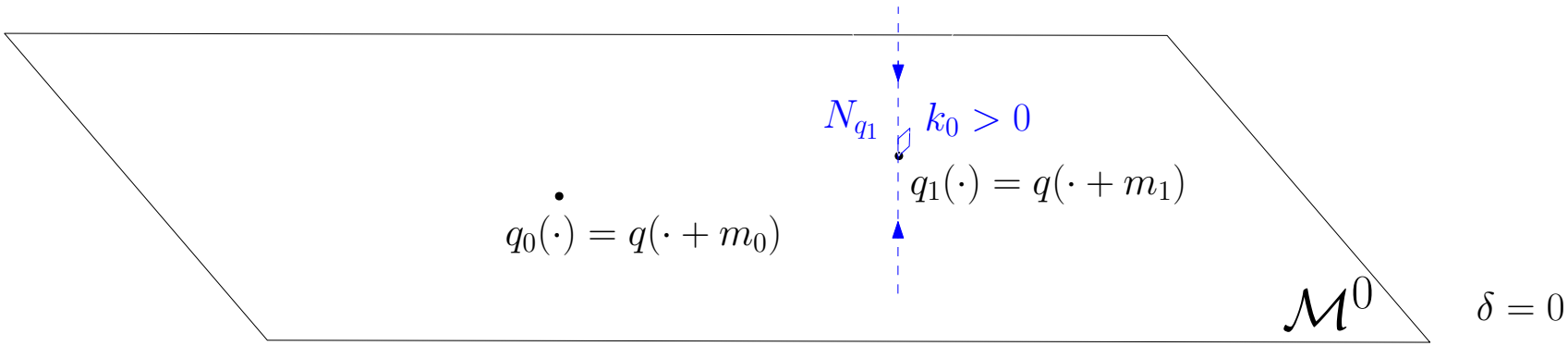
Perturbing normally hyperbolic invariant manifolds



Claim

- For $\delta = 0$, there is a trivial normally hyperbolic invariant (Gaussian) manifold for the evolution.
- Such structure is stable by small perturbations (Geometric singular perturbation theory : [Fenichel 1972], [Giacomin, Pakdaman, Pellegrin, Poquet 2012], [Wiggins 2013])

Perturbing normally hyperbolic invariant manifolds



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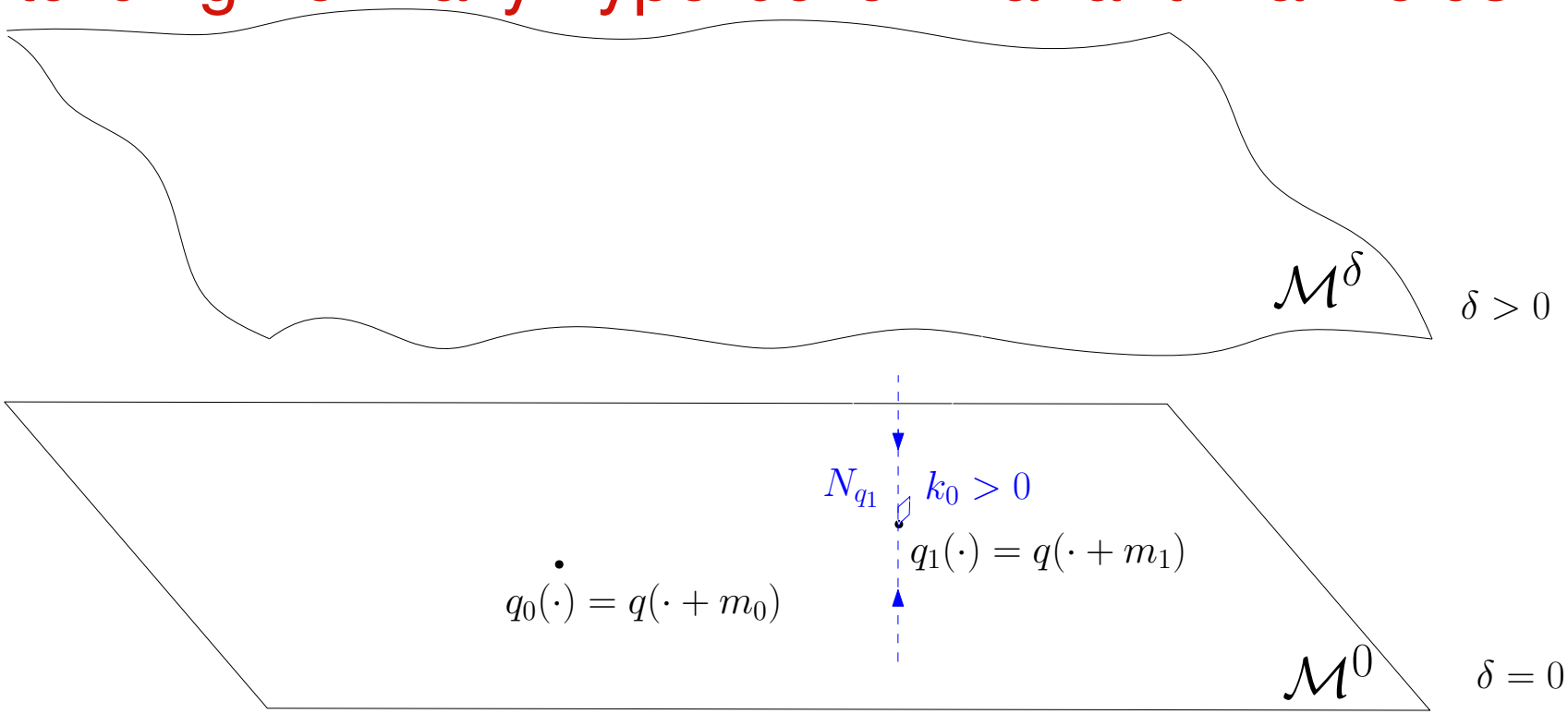

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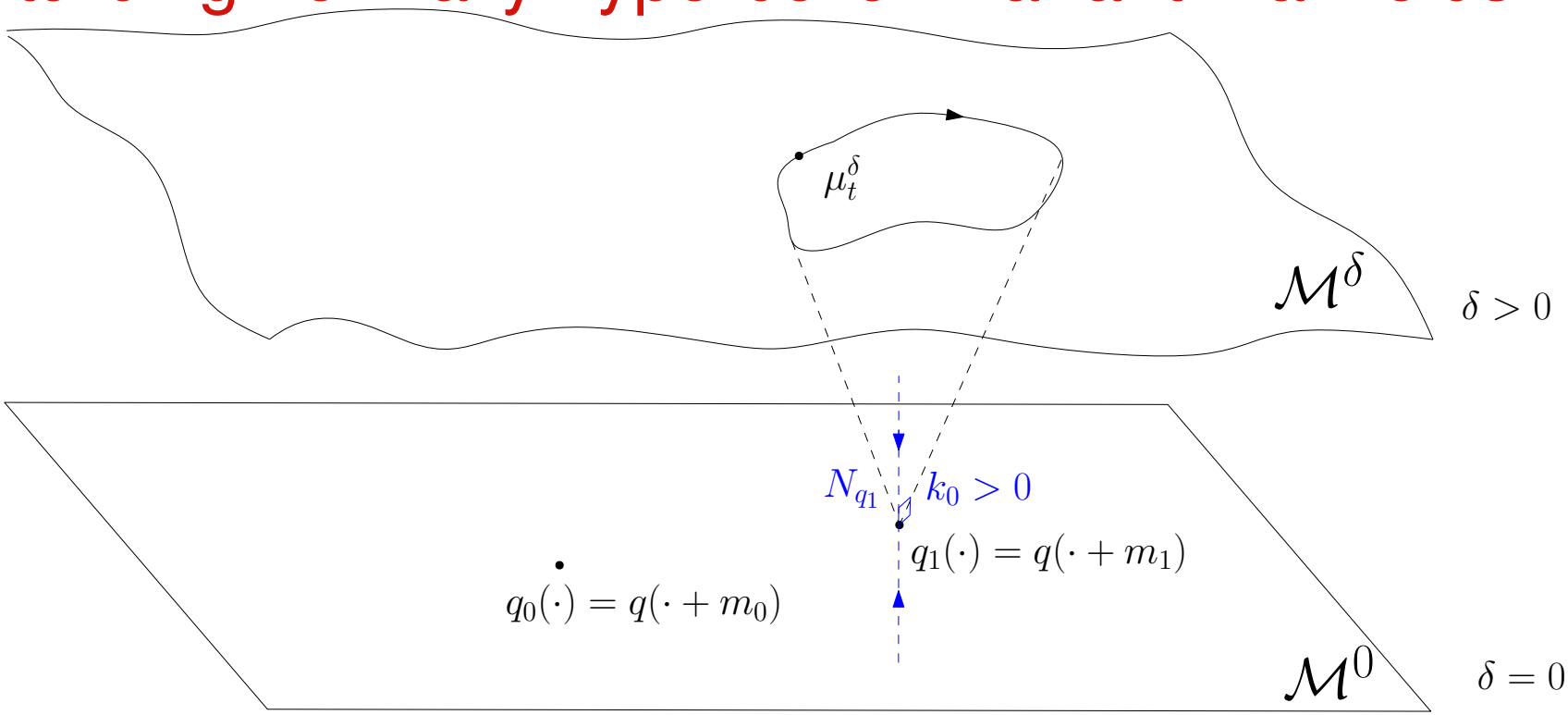

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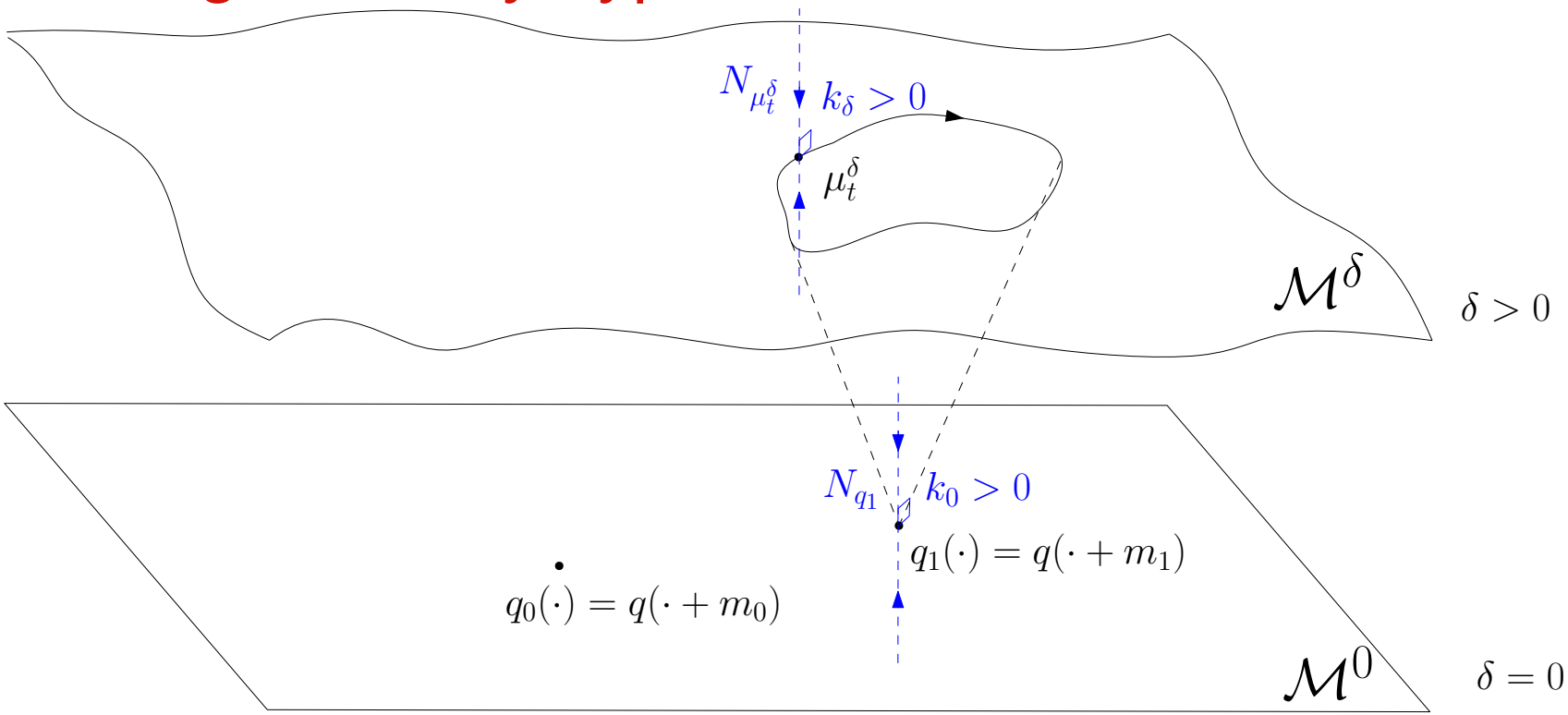

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- For $\delta = 0$, there is a trivial normally hyperbolic invariant (Gaussian) manifold for the evolution.
- Such structure is stable by small perturbations (Geometric singular perturbation theory : [Fenichel 1972], [Giacomin, Pakdaman, Pellegrin, Poquet 2012], [Wiggins 2013])

What is, at first order in δ, the dynamics of the mean-value m along the perturbed manifold ?

$$\left\{ \begin{array}{rcl} \partial_t p_t(x) & = & \nabla \cdot (\sigma^2 \nabla p_t(x)) + \nabla \cdot (p_t(x)(Kx + \dot{m}_t - \delta F(x + m_t))) \\ \dot{m}_t & = & \delta \int_{\mathbb{R}^d} F(x + m_t) p_t(\mathrm{d}x) \end{array} \right.$$

Intuitively,

- The fast component $p_t^\delta \approx p_t^{\delta=0}$ which converges exponentially fast to its invariant measure q.
- Hence, it is natural to replace p_t^δ by the Gaussian measure q in the slow component m.
- Thus, it is natural to expect

$$\dot{m}_t \; = \; \delta \int_{\mathbb{R}^d} F(x + m_t) q(x) \mathrm{d}x + O(\delta^2).$$

Theorem (L., Poquet, 2018)

There exists $\bar{\delta} > 0$ such that for all $0 \leqslant \delta \leqslant \bar{\delta}$, there exists a positively invariant manifold $\mathcal{M}^\delta = \left\{(p_m^\delta, m)\right\}_m$ for the nonlinear Fokker-Planck PDE, where p_m^δ is a probability measure on $\mathbb{R}^d$.

- *$\mathcal{M}^\delta$ is a perturbation of size δ of the Gaussian manifold $\mathcal{M}^0$*

$$\sup_m \|p_m^\delta - q\|_{L^2(w)} \leqslant C\delta.$$

- *$\mathcal{M}^\delta$ is stable in the following sense : if $\|p_0 - p_{m_0}^\delta\|_{L^2(w)} \leqslant c\delta$, then*

$$\|p_t - p_{m_t}^\delta\|_{L^2(w')} \leqslant ce^{-\lambda t}\|p_0 - p_{m_0}^\delta\|_{L^2(w')}$$

- *The trajectory $t \mapsto m_t^\delta$ of the mean-value of $p_{m_t}^\delta$ has the following expansion :*

$$\dot{m}_t^\delta = \delta \int_{\mathbb{R}^d} F(u + m_t)q(u)\mathrm{d}u + \delta^2 g^\delta(m_t^\delta),$$

with $\|g^\delta\| \leqslant C$.

To sum up

With this result, the dynamics of the infinite dimensional measure μ_t along the manifold $\mathcal{M}^\delta$ can be parameterized by the finite dimensional dynamics of its mean-value m_t^δ.
Namely, everything boils down to comparing now

- the dynamics of an isolated unit

$$\dot{m}_t = F(m_t) \tag{1}$$

- with the dynamics of the mean-value of connected units, approximated at first order in δ by

$$\dot{m}_t = \delta \int_{\mathbb{R}^d} F(x + m_t) q(x) \mathrm{d}x. \tag{2}$$

Claim

In the FHN case, for a suitable choice of parameters, Eq. (1) is in a resting state, whereas Eq. (2) oscillates : there is emergence of structured dynamics due to noise and interaction.

The FitzHugh-Nagumo model

Consider a FHN dynamics

$$F(v,w) = \left(v - \frac{v^3}{3} - w, \frac{1}{\tau}(v + a - bw)\right),$$

with diagonal interaction $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ and noise $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$.

Averaging F by the Gaussian kernel q gives

$$\int_{\mathbb{R}^2} F(z + (v,w)))q(z)\mathrm{d}z = \left(\left(1 - \frac{\sigma_1^2}{K_1}\right)v - \frac{v^3}{3} - w, \frac{1}{\tau}(v + a - bw)\right)$$

which is again a FitzHugh Nagumo system, where the parameter $u = 1$ has been changed into $u = 1 - \frac{\sigma_1^2}{K_1}$.

Summary

Adding noise and interaction simply means here decreasing the parameter u from 1 to $u = 1 - \frac{\sigma_1^2}{K_1} < 1$.

Bifurcations in the FitzHugh-Nagumo model [Rocsoreanu, Georgescu, Giurgiteanu, 2000]

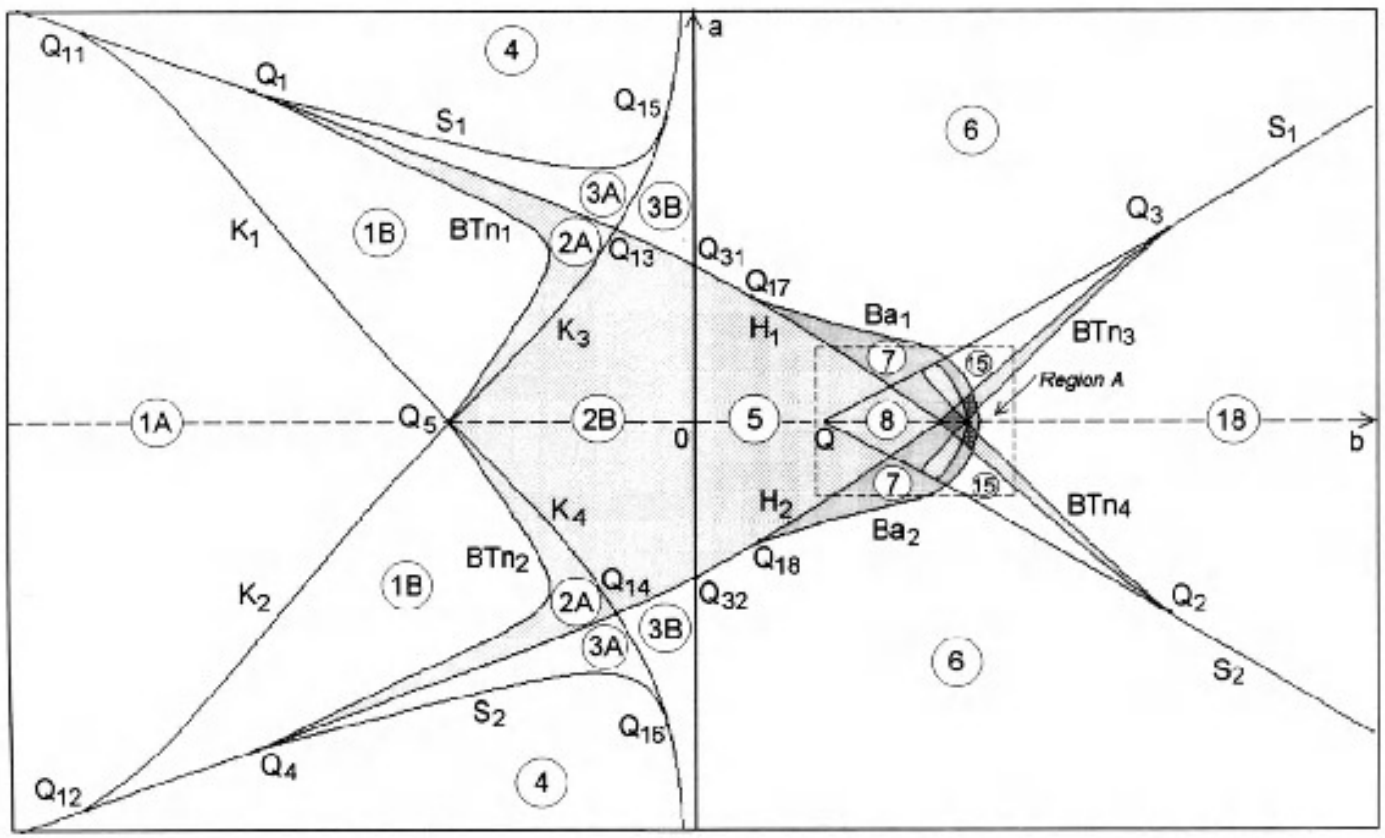
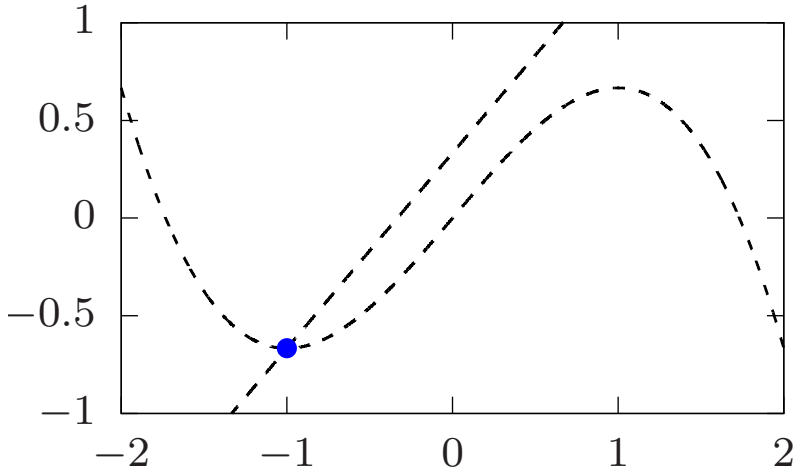


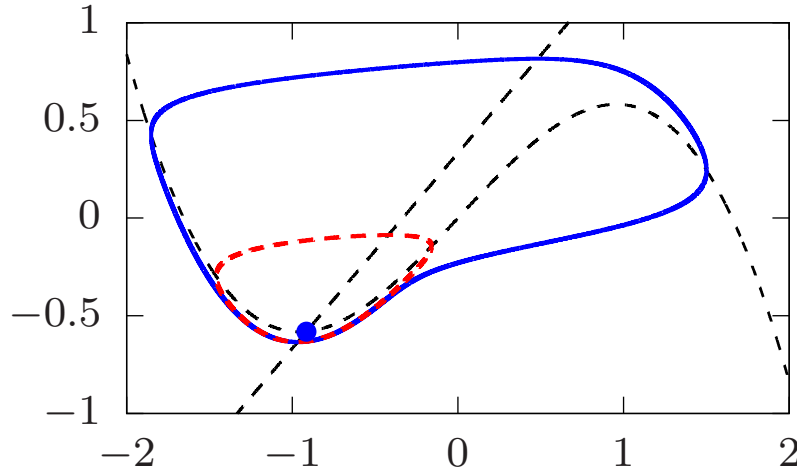
Fig.5.1.1a. Parametric portrait of the FitzHugh-Nagumo system.

Phase transition as $u = 1 - \frac{\sigma_1^2}{K_1} \searrow 0$ for $\dot{v} = uv - \frac{v^3}{3} - w, \dot{w} = \frac{1}{\tau}(v + a - bw)$

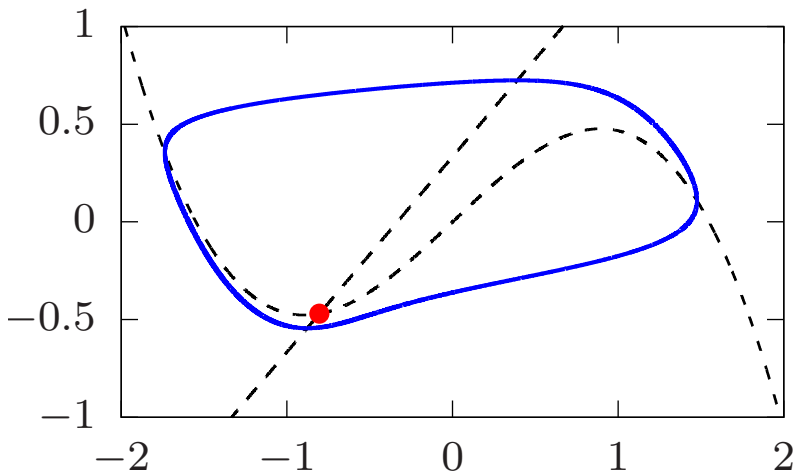
(a) $u = 1$



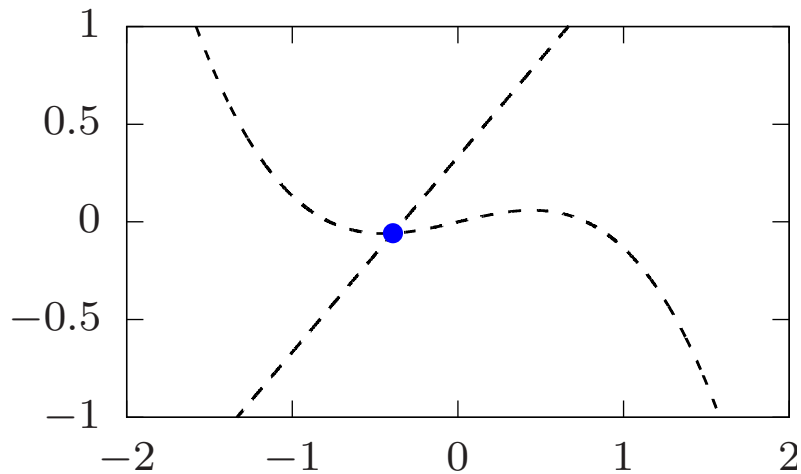
(b) $u = 0.914$



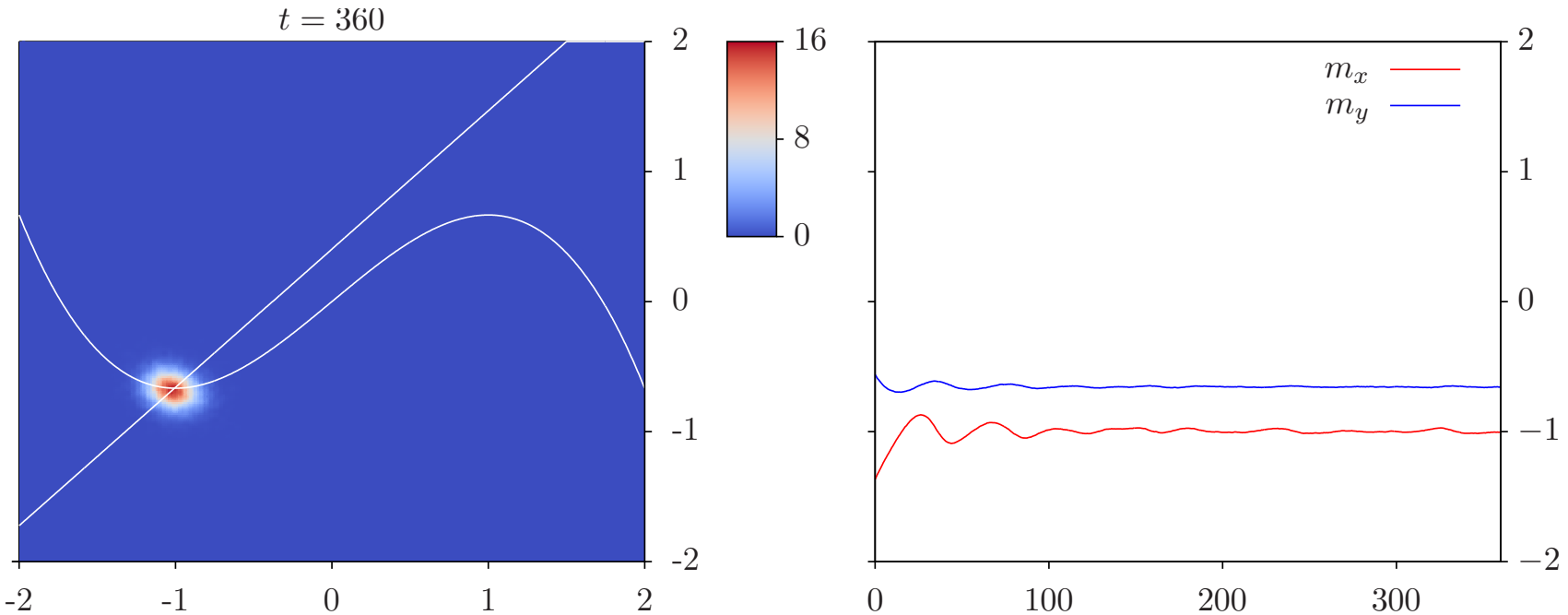
(c) $u = 0.8$



(d) $u = 0.2$



No noise : $\frac{\sigma_1^2}{K_1} = 0$ $(u = 1)$



Phase transition as $u = 1 - \frac{\sigma_1^2}{K_1} \searrow 0$ for $\dot{v} = uv - \frac{v^3}{3} - w, \dot{w} = \frac{1}{\tau}(v + a - bw)$

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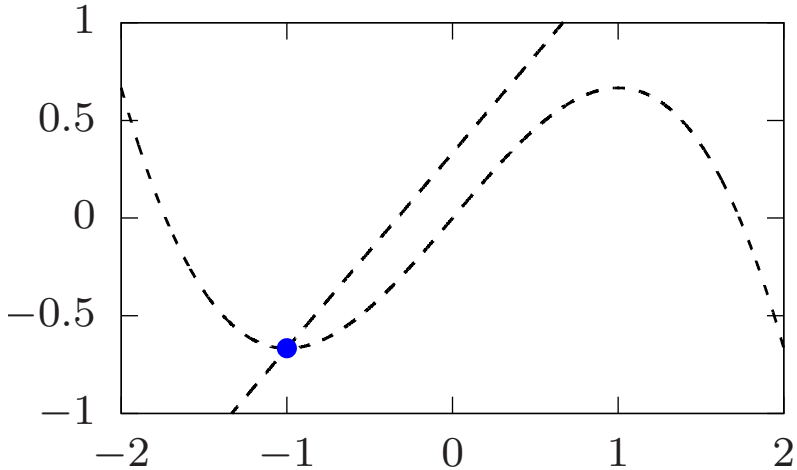

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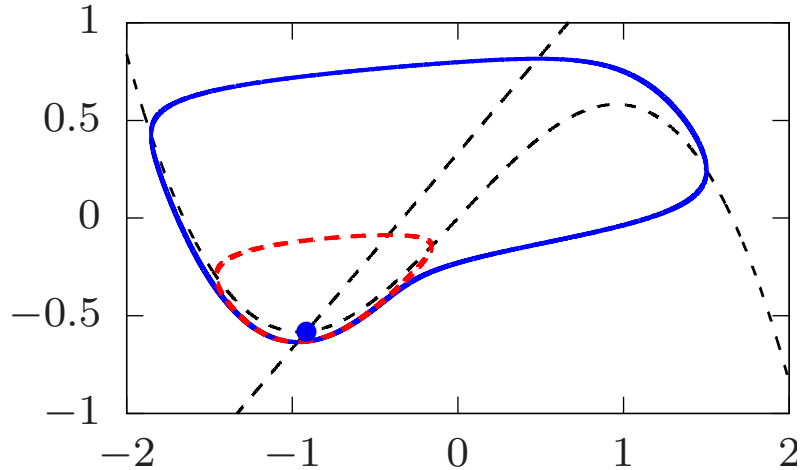

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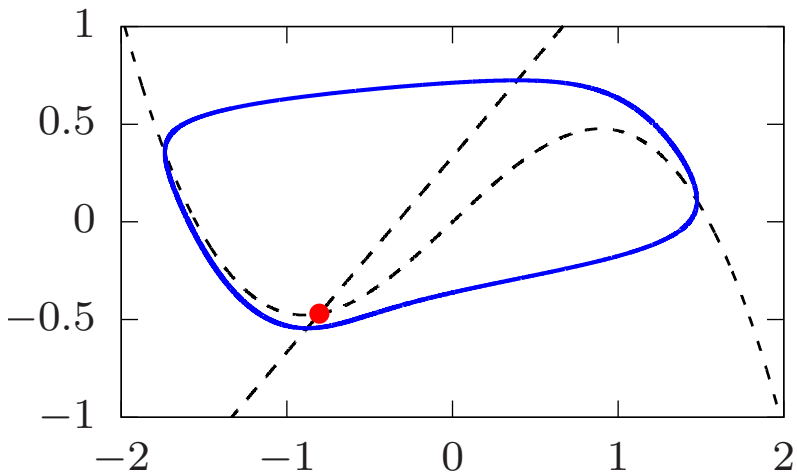

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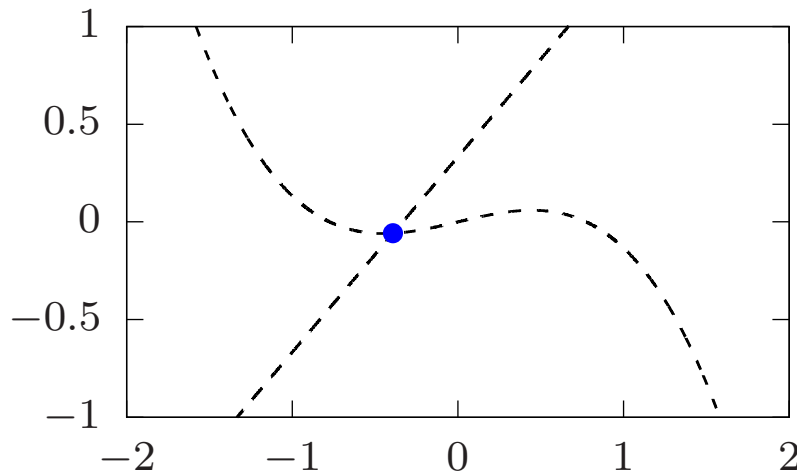
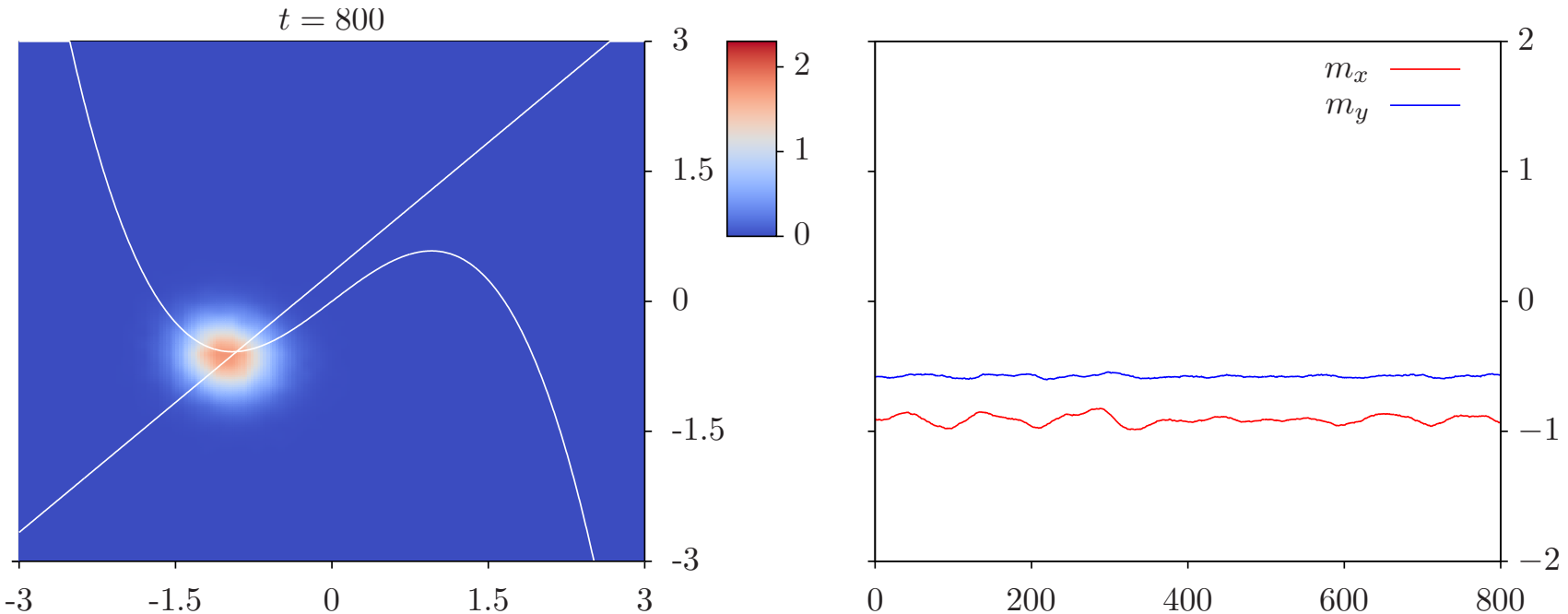
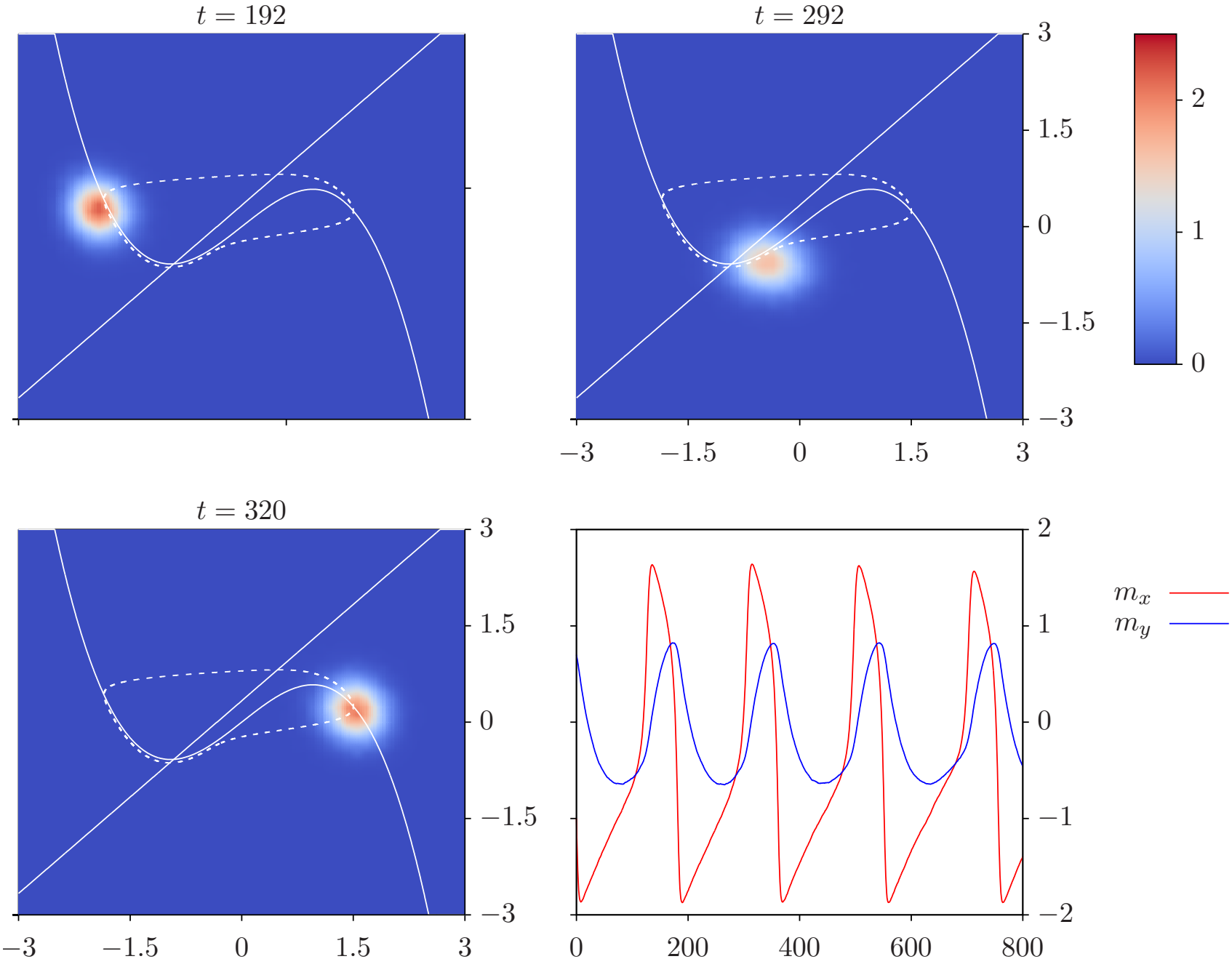


Small noise : $\frac{\sigma_1^2}{K_1} = 0.086$ $(u = 0.914)$: coexistence of resting point and limit cycle




$t = 192$
$t = 292$
$t = 320$
3
1.5
0
−1.5
−3
2
1
0
−3
−1.5
0
1.5
3
2
1
0
−1
−2
0
200
400
600
800
m_x
m_y

Phase transition as $u = 1 - \frac{\sigma_1^2}{K_1} \searrow 0$ for $\dot{v} = uv - \frac{v^3}{3} - w, \dot{w} = \frac{1}{\tau}(v + a - bw)$

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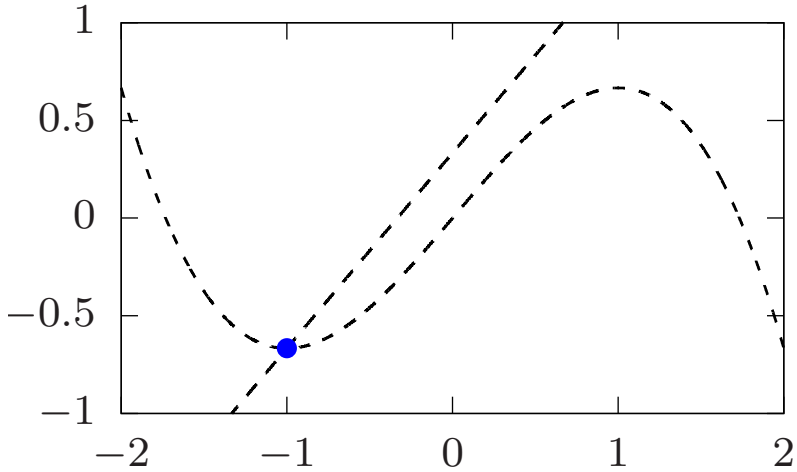
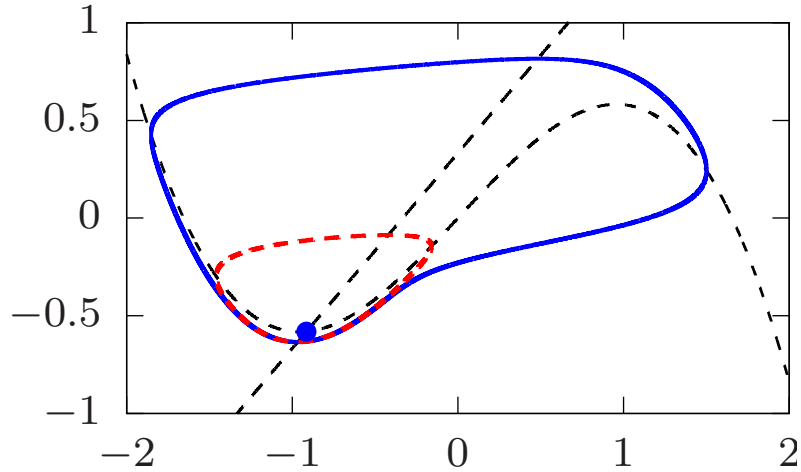
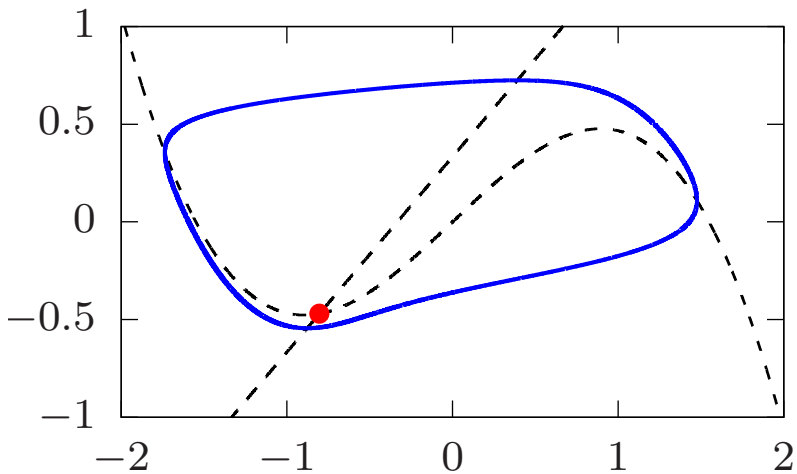


(b) $u = 0.914$



(c) $u = 0.8$



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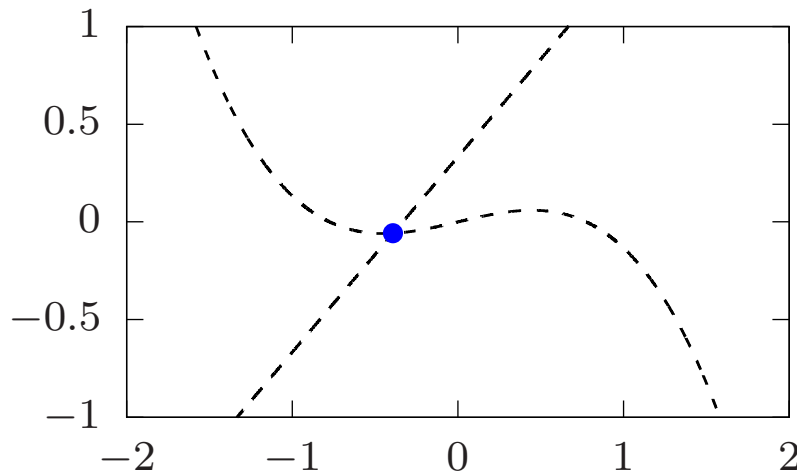
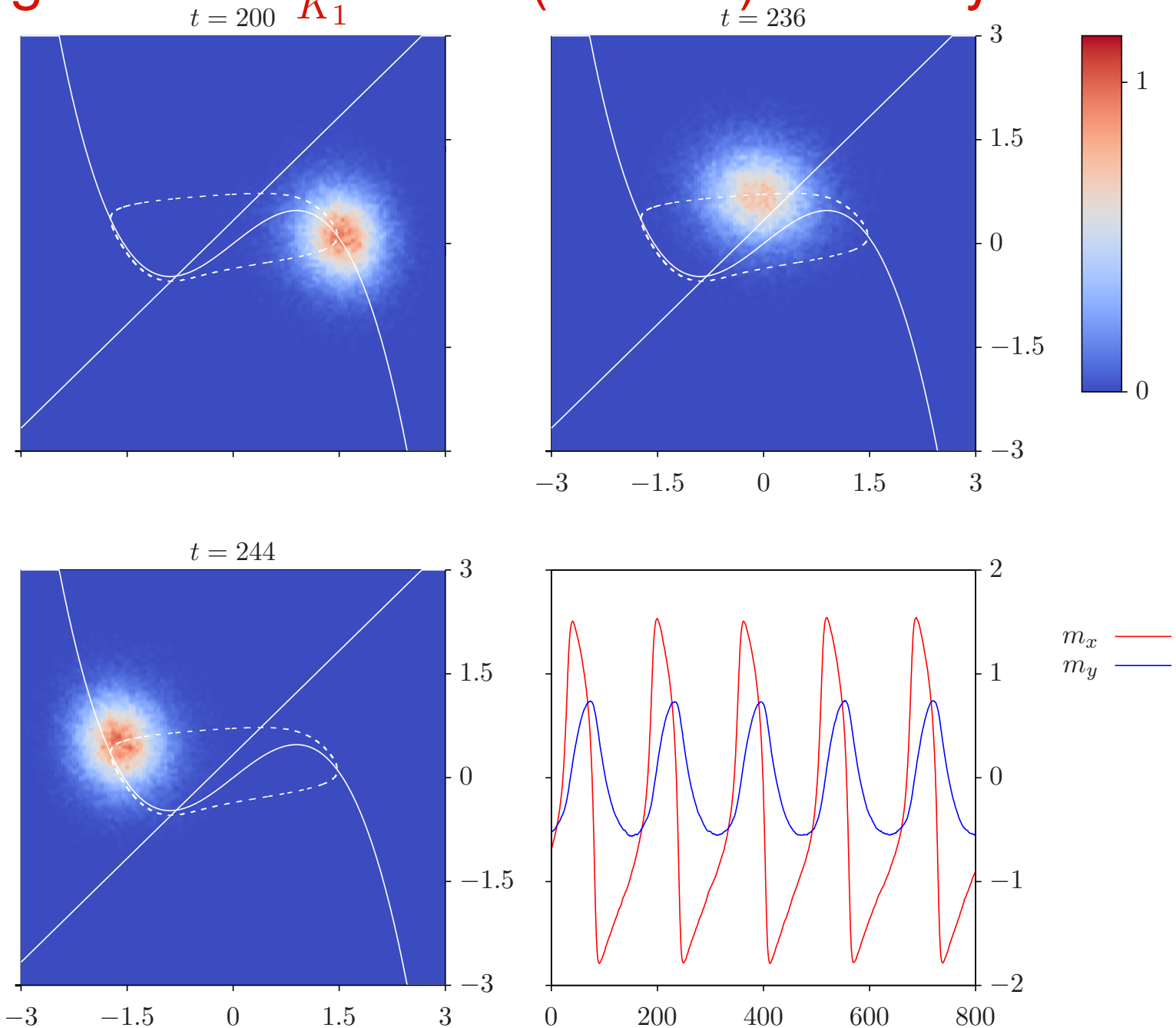
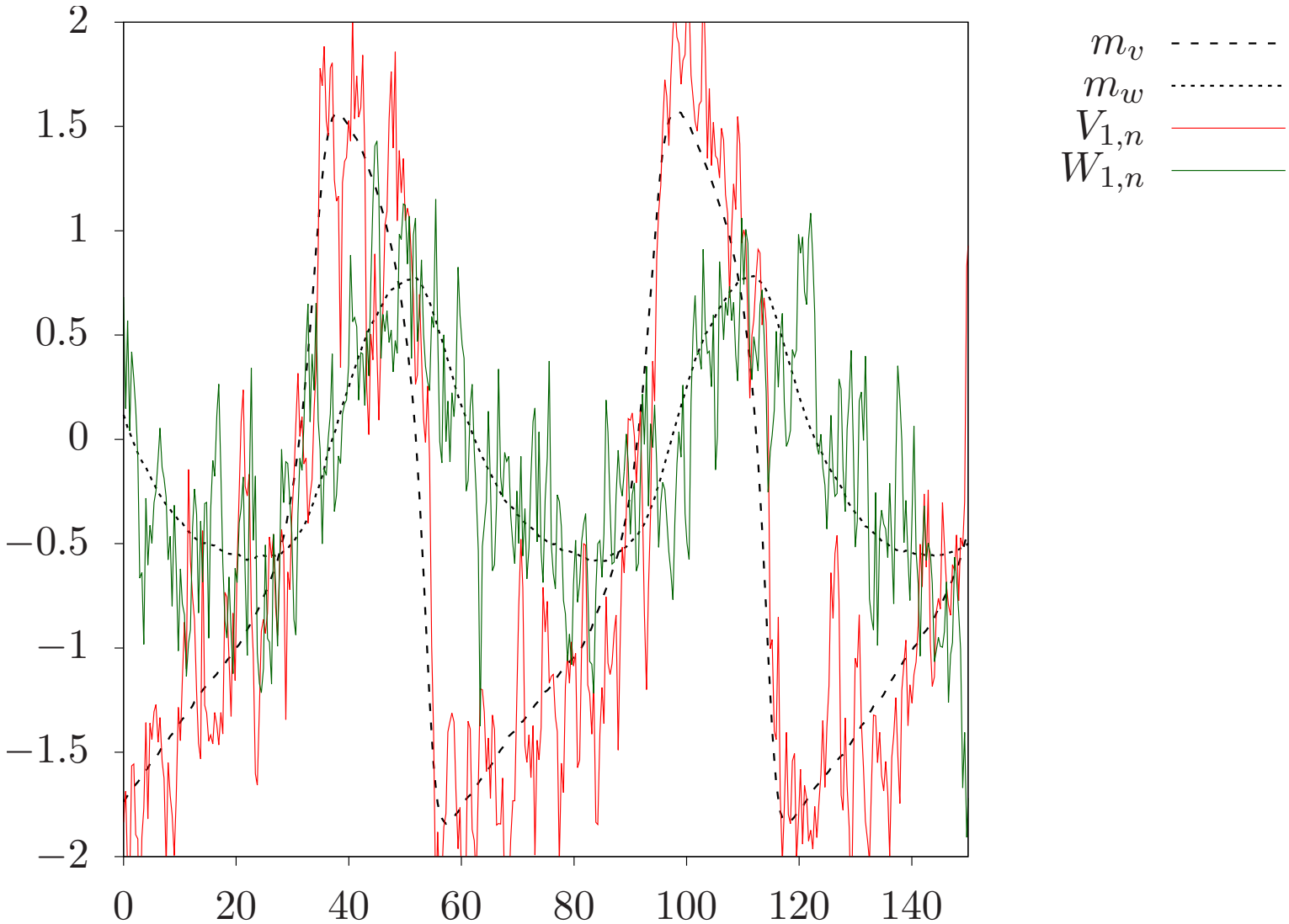


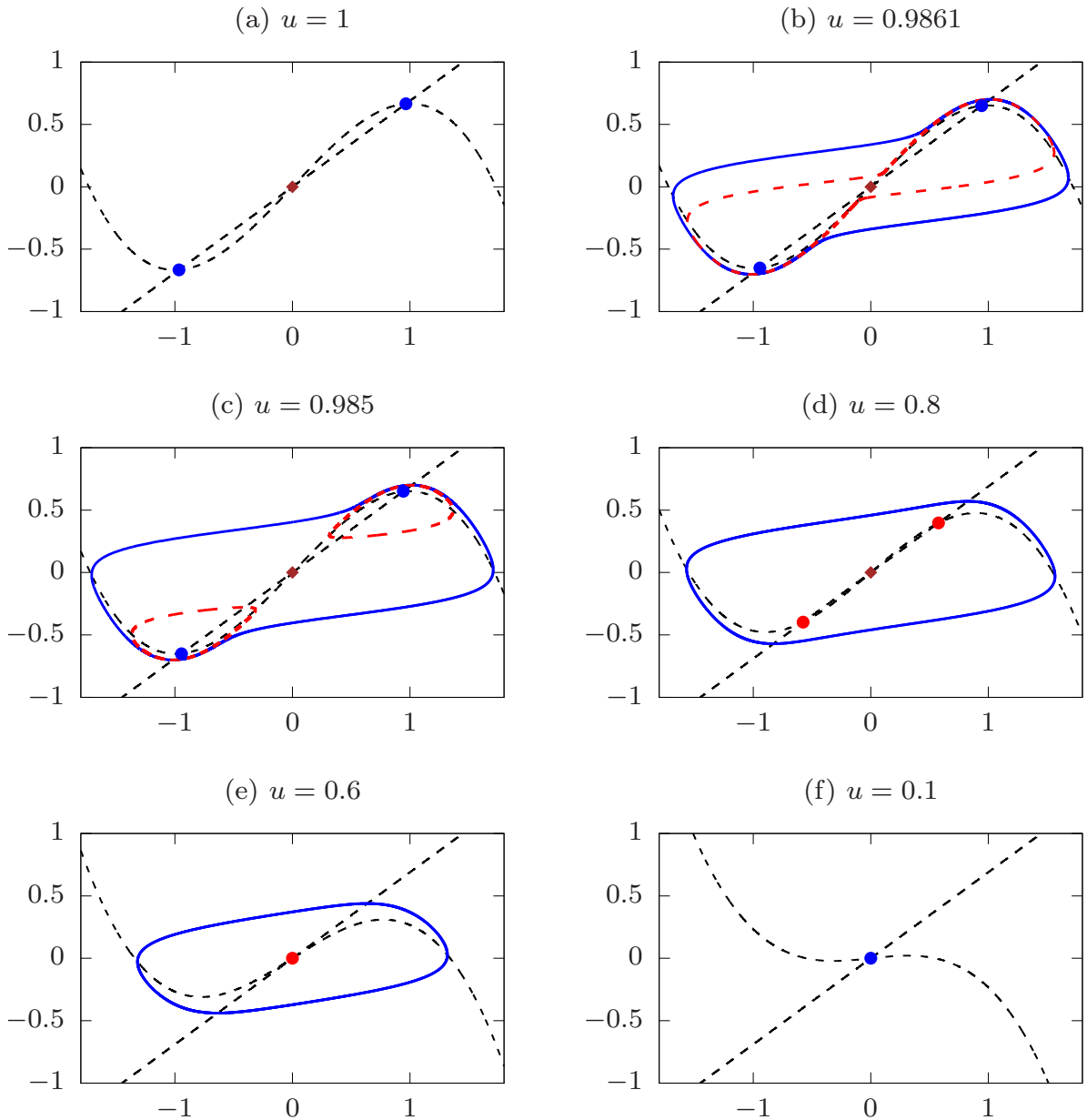
Large noise : $\frac{\sigma_1^2}{K_1} = 0.2$ $(u = 0.8)$: limit cycle

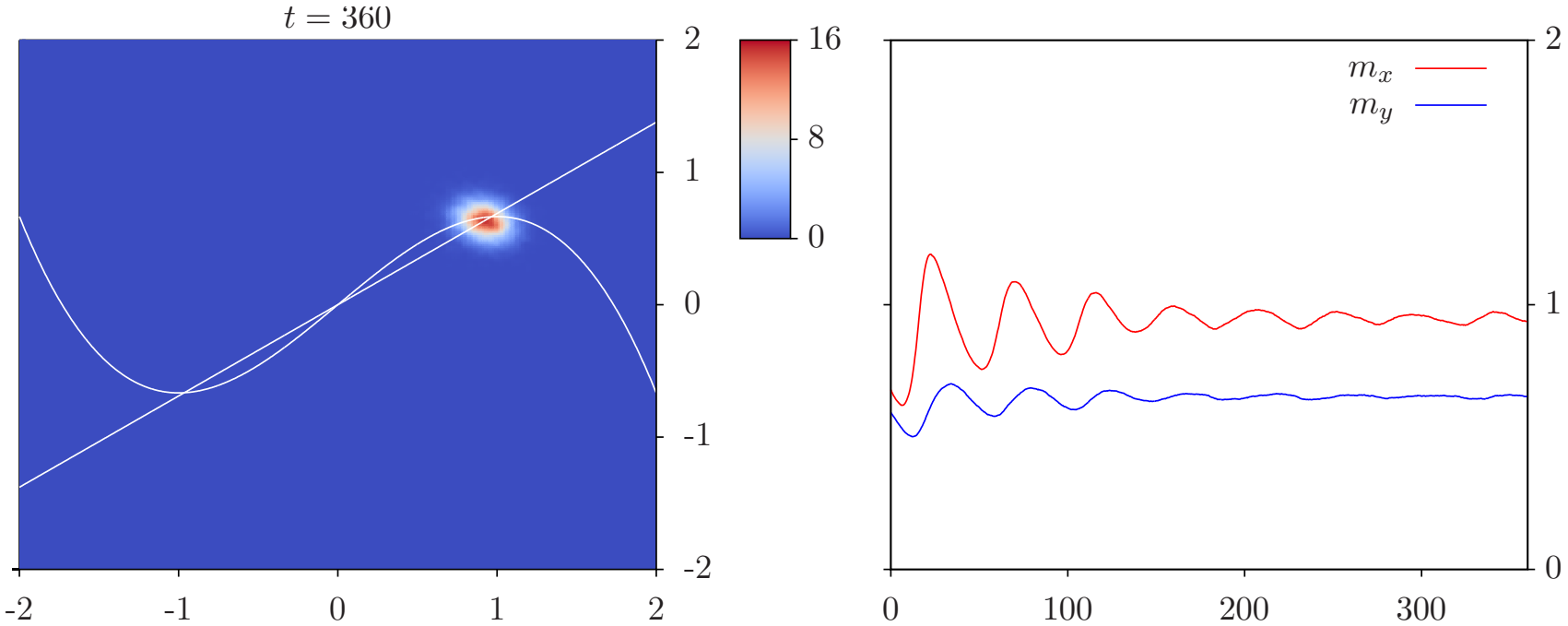


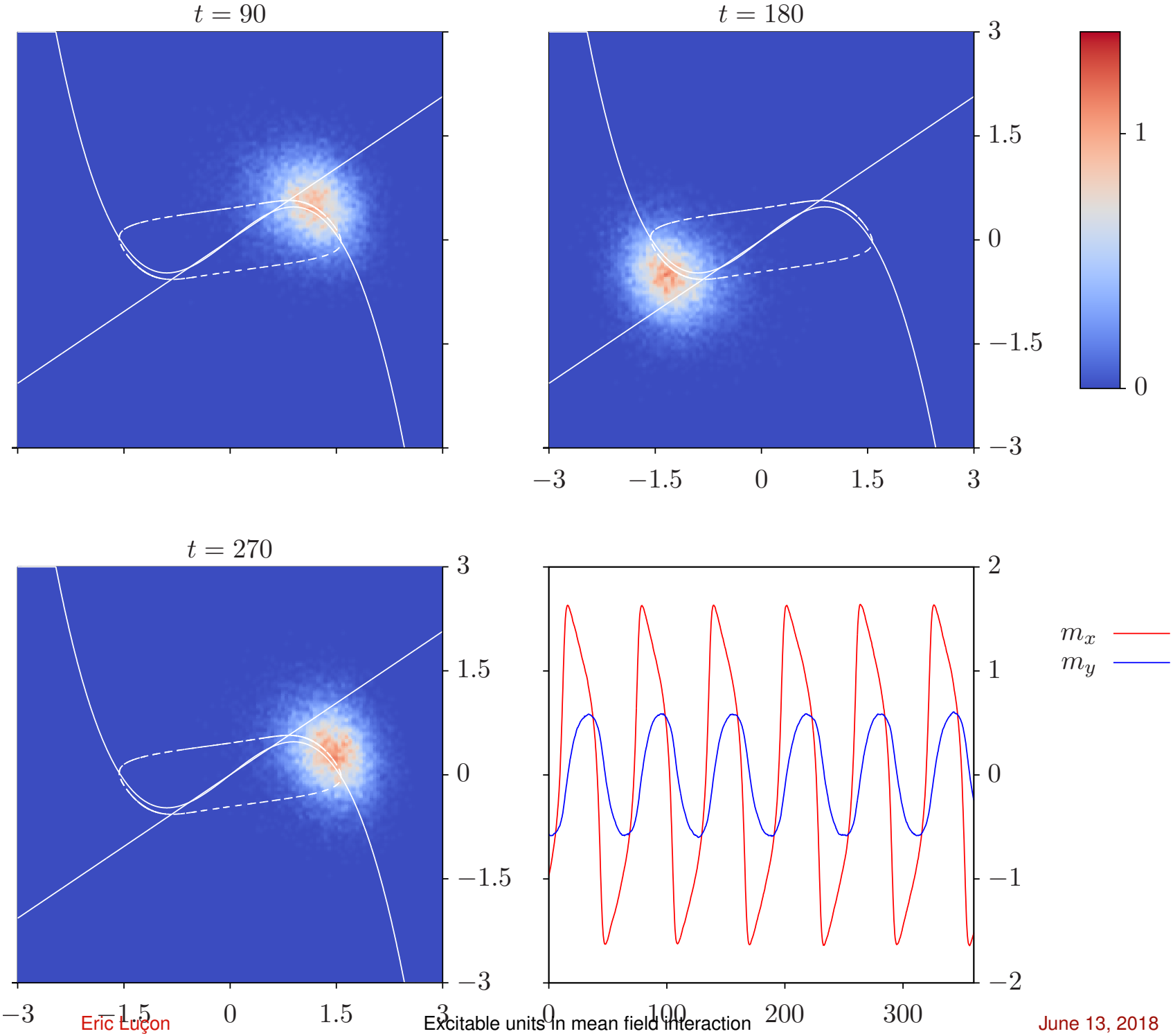
Behavior of one single particle



The bistable case : phase transition as $u = 1 - \frac{\sigma_1^2}{K_1} \searrow 0$



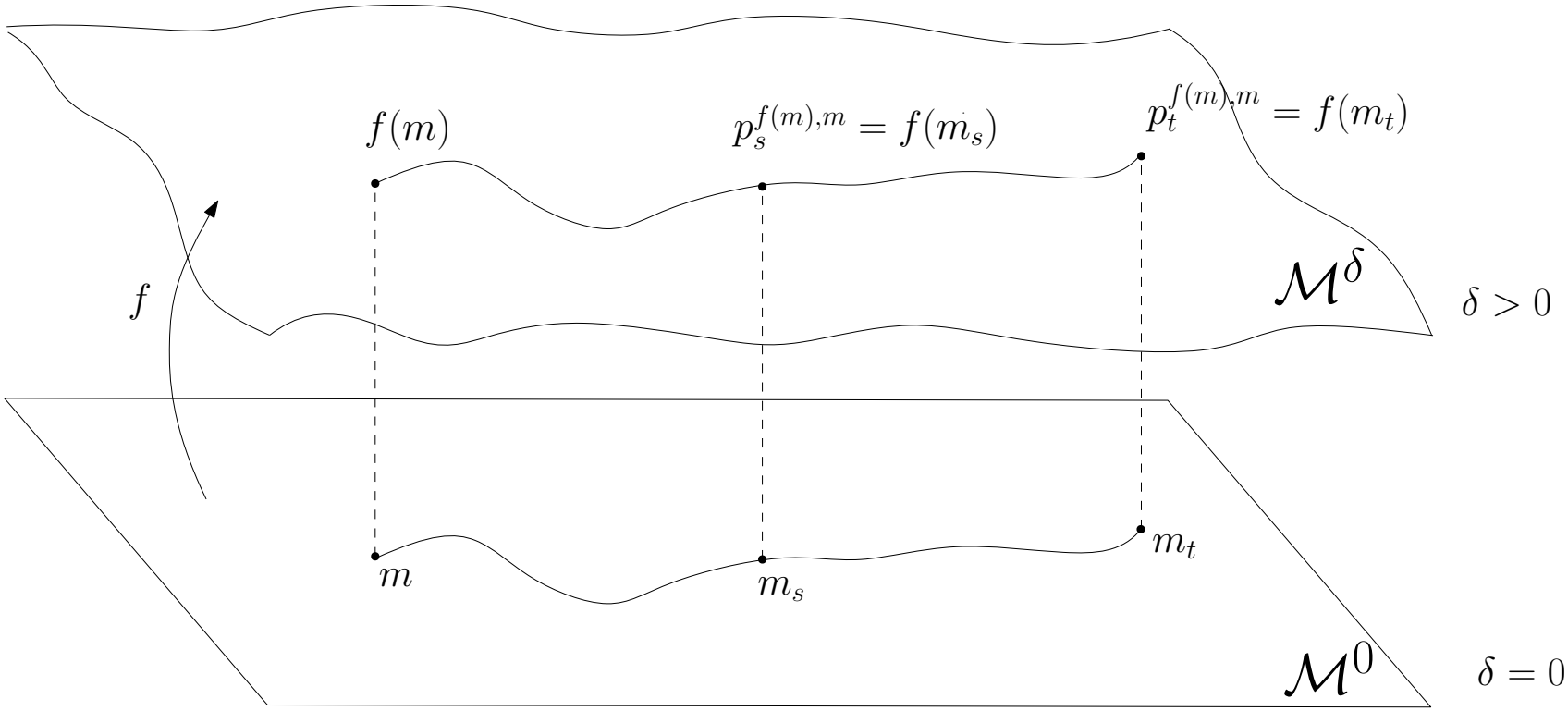

$t = 360$
2
1
0
-1
-2
-2
-1
0
1
2
16
8
0
m_x
m_y
2
1
0
0
100
200
300


$t = 90$
$t = 180$
$t = 270$
3
1.5
0
−1.5
−3
1
0
2
−1
−2
100
200
300
m_x
m_y

Thank you for your attention !

Emergence of oscillatory behaviors for excitable systems with noise and mean-field interaction, a slow-fast dynamics approach, E. Luçon, C. Poquet, arXiv :1802.06410.

Idea of proof : fixed-point argument



Fixed-point problem

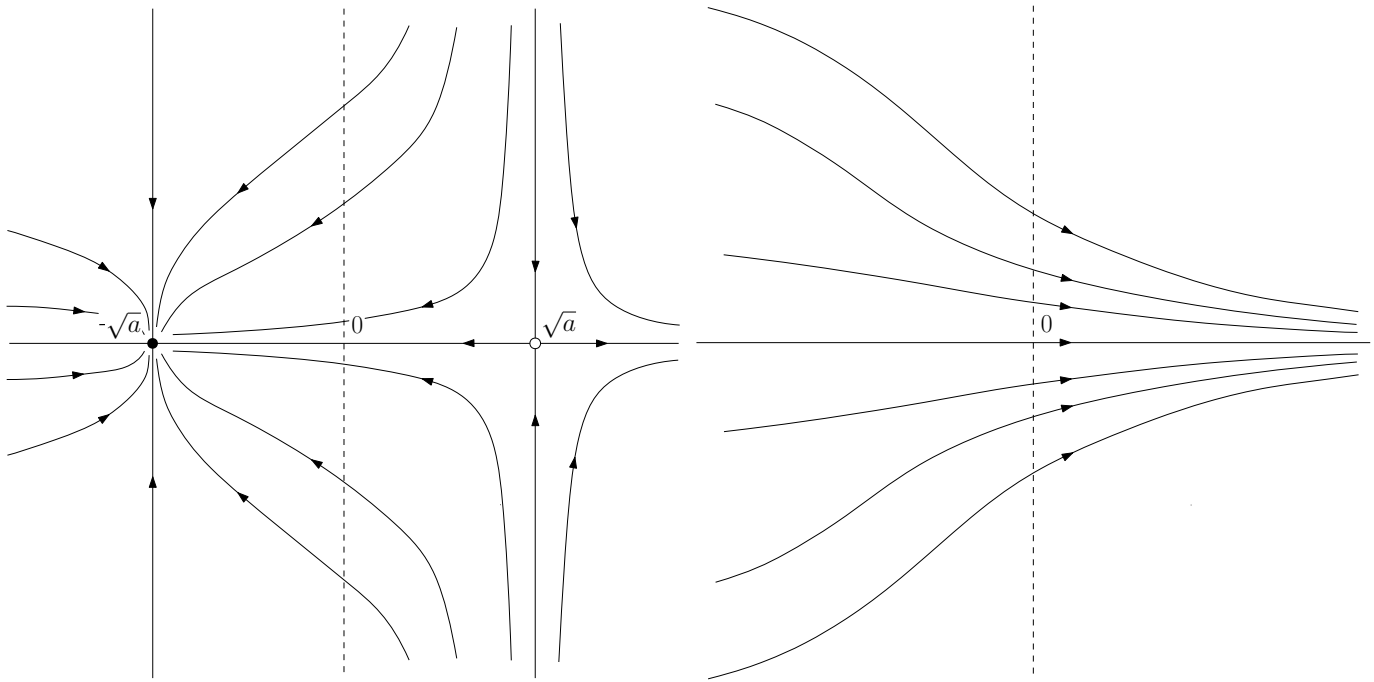
If $p_t^{p_0,m_0}$ is the solution of the PDE with initial condition p_0 with mean m_0, find f such that

$$f(m_t) = p_t^{f(m),m}, \ t \geqslant 0.$$

A toy-model (I)

$$F(x,y) \;=\; \left(x^2 - a, -by\right) \;,$$

- $F = -\nabla V$ with $V(x,y) = ax - \frac{x^3}{3} + \frac{by^2}{2}$,
- Simple prototype of a dynamics with a saddle-node bifurcation.



(a) $a > 0$ (b) $a < 0$

A toy-model (II)

$$F(x,y) \;=\; \left(x^2 - a, -by\right) \;,$$

The averaged functional becomes

$$\int_{\mathbb{R}^2} F(u) q_{(x,y),\frac{\sigma^2}{K}}(u)\mathrm{d}u \;=\; \left(x^2 - a + \frac{\sigma^2}{K}, -by\right) \;. \tag{3}$$

Now the bifurcation reads in terms of $\frac{\sigma^2}{K} < a$ and $\frac{\sigma^2}{K} > a$:

- for $\frac{\sigma^2}{K} < a$: the noise is not strong enough, no noise-induced behavior.
- for $\frac{\sigma^2}{K} > a$: the noise is sufficient to make the system drift to the right.

Stuart-Landau Model ($a > 0$, $\omega \in \mathbb{R}$)

$$F(x,y) = \left(x(a-(x^2+y^2)) - \omega y, y(a-(x^2+y^2)+\omega x)\right) ,$$

In polar coordinates,

$$\begin{cases} \dot{r} = r(a-r^2) \\ \dot{\theta} = \omega \end{cases}$$

which admits the stable limits cycle $r_t = \sqrt{a}$, $\theta_t = \theta_0 + \omega t$.
The averaged system is again a Stuart Landau model : for $\varpi = \frac{\sigma^2}{K}$,

$$\begin{aligned} &\int_{\mathbb{R}^2} F(u) q_{(x,y),\varpi}(u) \mathrm{d}u \\ &= \left(x(a-4\varpi^2-(x^2+y^2)) - \omega y, y(a-4\varpi^2-(x^2+y^2)+\omega x)\right) . \end{aligned} \tag{4}$$

First observation : if $a - 4\varpi^2 > 0$, then we have persistence of a limit cycle.

Modified Stuart-Landau model

$$F(x,y) = \begin{pmatrix} x(1-(x^2+y^2)) - (\omega - by)y \\ y(1-(x^2+y^2)) + (\omega - by)x\,. \end{pmatrix}$$

which gives, in polar coordinates

$$\begin{cases} \dot{r} = r(1-r^2), \\ \dot{\theta} = w - br\sin\theta \end{cases} \tag{5}$$

When $\omega > 0$ and $b > \omega$ but close to ω : simple example of excitable dynamics in $\mathbb{R}^2$: $(r_0, \theta_0) = (1, \arcsin(\omega/b))$ is a stable fixed-point, and $(1, \pi - \arcsin(\omega/b))$ is unstable.

The averaged system

The averaged system becomes, for $\varpi = \frac{\sigma^2}{K}$

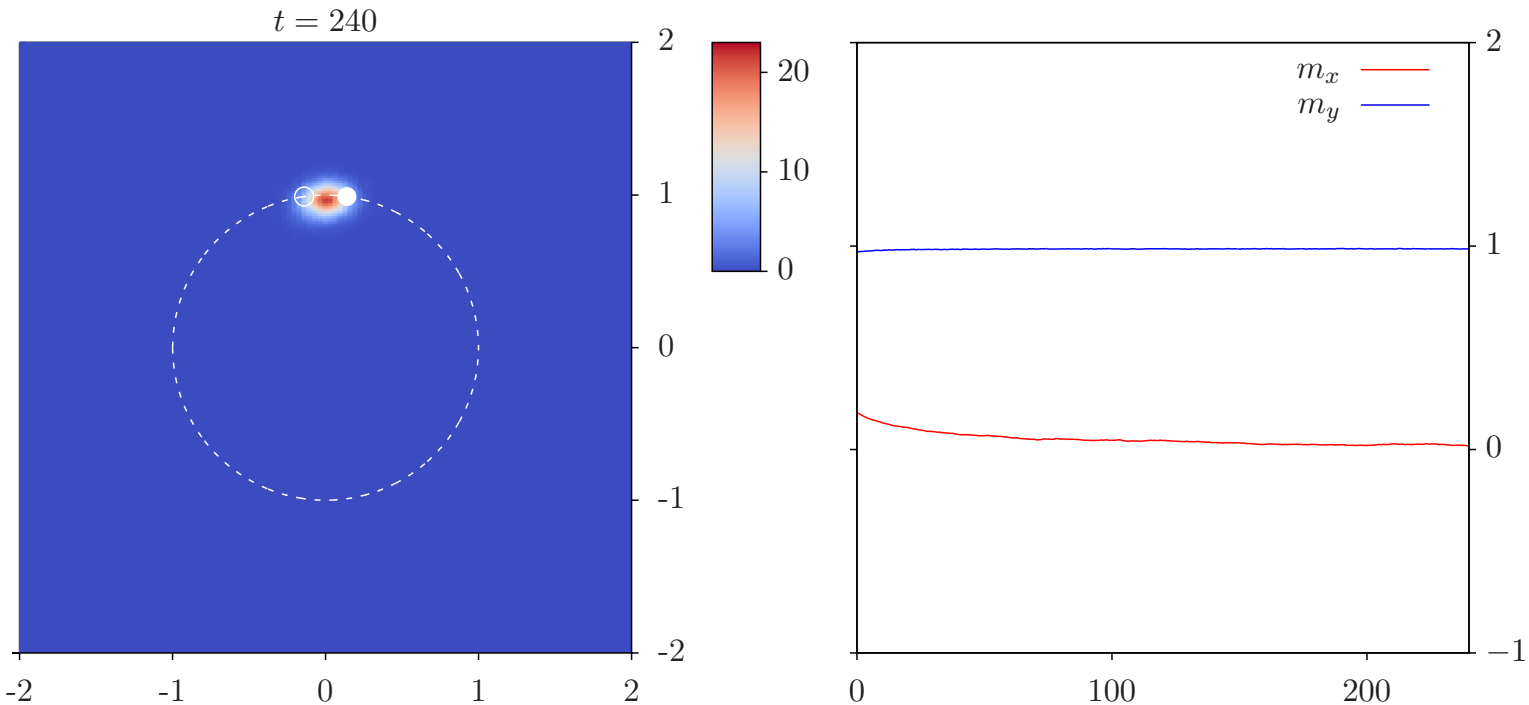
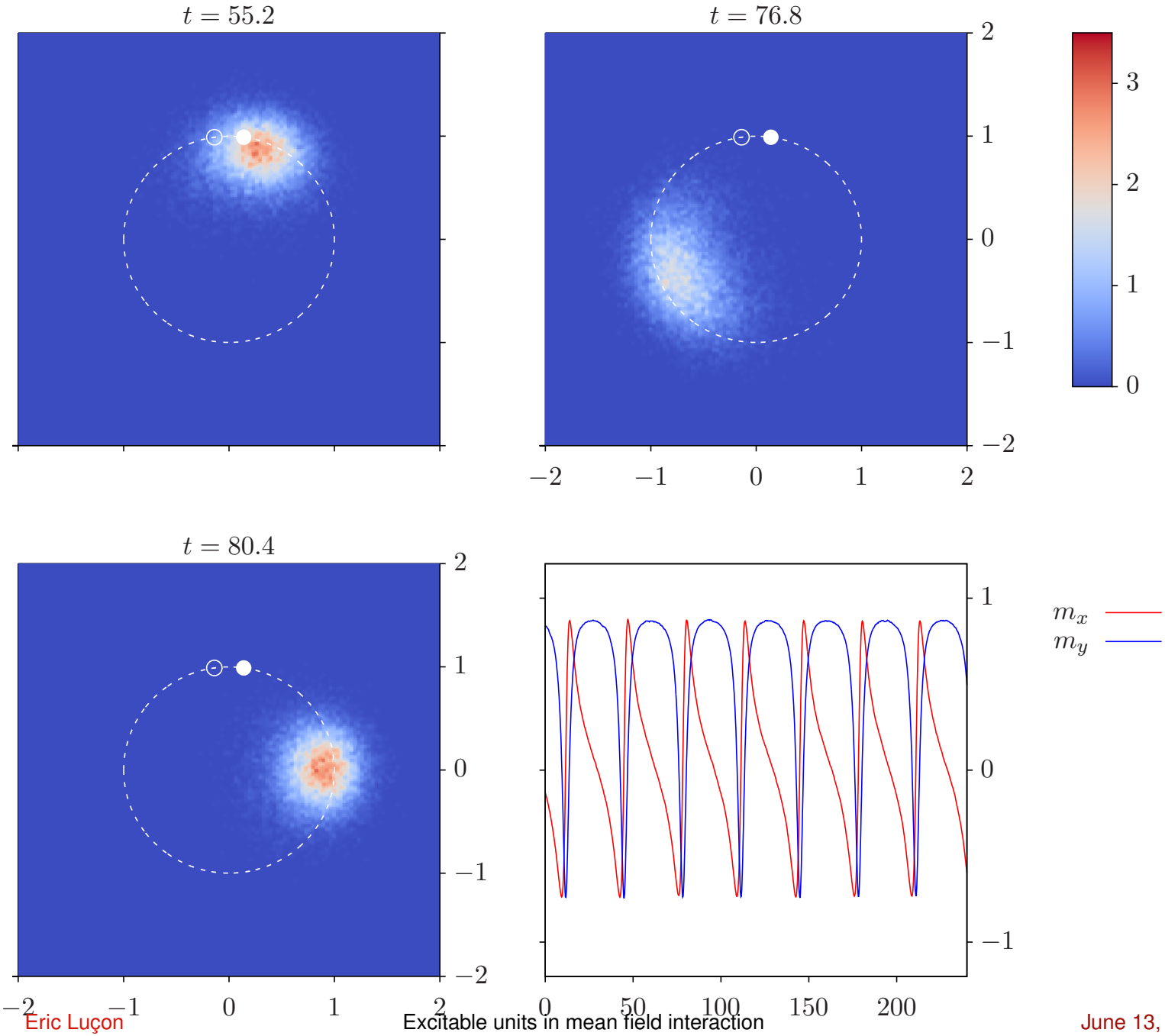
$$\begin{cases} \dot{r} = r\left(1 - 4\varpi - r^2\right) - b\varpi\cos\theta, \\ \dot{\theta} = \omega - b\left(r - \varpi\frac{1}{r}\right)\sin\theta\,. \end{cases} \tag{6}$$

Suppose now that $\omega = b(1-\zeta)$ with ζ small. If ζ and ϖ are small enough the invariant manifold $\{r = 1\}$ persists under perturbation, becoming an invariant curve $\{r_c(\theta)\}$, which satisfies for ϖ small enough

$$0 \;\leqslant\; r_c(\theta) - \varpi\frac{1}{r_c(\theta)} \;\leqslant\; 1 - \frac{1}{2}\varpi\,. \tag{7}$$

We deduce the existence of a limit circle if $1 - \frac{1}{2}\varpi < 1 - \zeta$, i.e. ϖ is small enough and satisfies $\varpi > 2\zeta$.

Small noise : fixed-point



Large noise : limit cycle