Mean-field limits for particle systems with excitatory feedback

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We consider a McKean–Vlasov problem arising as the mean-field limit of a particle system with an excitatory feedback effect: As particles hit a barrier, they cause the other particles to jump in the direction of the barrier and this feedback mechanism leads to the possibility that the system can exhibit contagious blow-ups. While our motivation comes from the study of contagion in financial networks (i.e., financial crises), the problem turns out to be analogous to recent mean-field models for self-excitatory networks of electrically coupled neurons within a nonlinear noisy integrate-and-fire framework [3, 4, 6, 7, 10].

In [3] it is shown that the neuroscience version of the problem has a unique smooth solution when the feedback effect is sufficiently small, however, it is also known that the system must exhibit blow-ups if the feedback parameter is too large (see [7, 2]). Moreover, it is known from [6] that there is global well-posedness for any feedback strength if one considers instead a certain mollification of the feedback effect, which amounts to a non-instantaneous transmission of the feedback (because it is transmitted along the dendrites according to a cable equation).

Our contributions here are twofold and concern new developments related to both [3] and [6], where the former can be naturally viewed as a limiting case of the latter — a point of view that underlines the importance of both models.

Firstly, we study the model with instantaneous feedback (analogous to [3]). In this case, we establish a new uniqueness result up to the first blow-up time in the class of càdlàg functions, which confirms the validity of related propagation-of-chaos results. Moreover, we extend the allowed initial conditions to take us asymptotically close to a global uniqueness theory, for which precise conjectures are provided.

Secondly, we establish a global existence and uniqueness theory in the non-instantaneous setting under the addition of a common source of noise. In this way, the density of the mean-field limit becomes stochastic and the qualitative behavior of the system now depends intimately on the realisations of the common noise. This is a very natural extension given our motivations from mathematical finance, but we have also seen examples of its relevance in the computational neuroscience literature — e.g. the two-neuron model in [8].

The above results are the subject of two recent preprints [1, 2]. Also, it is worth emphasizing that our results complement similar results for models based on self-exciting point processes, see e.g. [5, 9].

References

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