Emergence of collective behavior for excitable units in mean-field interaction.

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Excitable systems are ubiquitous in the modeling of neural activity: one neuron that would stay in a resting state in the absence of any stimuli would generate action potentials in the presence of a sufficiently large input. This general phenomenon which has been observed physiologically and confirmed for many usual neuronal models (e.g. Morris-Lecar or FitzHugh-Nagumo models) is the subject of a vast literature, especially at the level of the dynamics of one single neuron. In this context, the presence of thermal noise appears to have a significant influence (see [2]).

Fewer rigorous results (see [5]) exist in a similar situation of a large population of excitable units in the presence of noise and interaction (the main difficulty of the analysis of such excitable systems being their absence of reversibility).

The point of this work [3] is to address the dynamics of excitable units in the limit of large population, in the presence of noise and interaction. A general framework for the analysis is given by the following system of $N \gg 1$ interacting diffusions with linear interaction

$$\mathrm{d}x_{i,t} = \Big(\delta F(x_{i,t}) - K\big(x_{i,t} - \frac{1}{N}\sum_{j=1}^{N} x_{j,t}\big)\Big)\mathrm{d}t + \sqrt{2}\sigma \mathrm{d}B_{i,t},\ i = 1, \ldots, N,$$

where $\sigma > 0$ tunes the intensity of the Gaussian noise $B_{i,t}$, $K > 0$ captures the intensity of the interaction and $F(\cdot)$ is a general intrinsic dynamics for a single neuron. In the limit of $N \to \infty$, the system is well-described by the density of particles $t \mapsto \mu_t$, solution to the following nonlinear Fokker-Planck equation (see also [5, 4])

$$\partial_t \mu_t = \sigma^2 \Delta \mu_t + K \nabla \cdot \left(\mu_t \left(x - \int_{\mathbb{R}^d} z \mu_t(\mathrm{d}z) \right) \right) - \delta \nabla \cdot (\mu_t F(x)),\ t \geq 0\,. \tag{1}$$

The aim of this work is to address the long-time behavior of (1) for a general class of dynamics $F(\cdot)$, that includes the cases of FitzHugh-Nagumo or Stuart-Landau oscillators.

Our main theorem concerns the existence of a stable invariant manifold for the Fokker-Planck PDE (1) in a regime of small dynamics ($\delta \ll \frac{K}{\sigma^2}$). A byproduct of the result is an explicit parameterization of the dynamics of (1) on the perturbed manifold in terms of the mean value of the density μ_t. In the particular case of Stuart-Landau units, this result illustrates rigorously the necessity of noise and interaction for the appearance of oscillatory behaviors in mean-field systems (coherent resonances). For FitzHugh-Nagumo oscillators, this result shows the stability of periodic behaviors under noise and interaction.

This results relies on perturbation techniques for normally contracting manifolds already used in the case of phase oscillators (Active Rotators model [1]).

References

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