Traveling fronts for lattice neural field equations

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In this study, we consider the following lattice differential equation

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) = -u_n(t) + \sum_{j\in\mathbb{Z}} K_j S(u_{n-j}(t)), \quad t > 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $u_n(t)$ represents the membrane potential of neuron labelled n at time t. Here K_j represents the strength of interactions associated to the neural network at position j on the lattice and the firing rate of neurons $S(u)$ is a nonlinear function. Such an equation can be seen as a Hopfield neural network model with infinite range interactions [6] or more simply as a discrete neural field equation [2] where each neuron is set on the lattice $\mathbb{Z}$ with all to all couplings. We report on two results [4] regarding the lattice neural field equation (LNFE) (1).

Result 1: We show existence and uniqueness of traveling front solutions for the LNFE (1) in the regime where the kinetics of each individual neuron is of bistable type. That is when $f(u) = -u + S(u)$ has three zeroes, namely $u = 0$, $u = \theta \in (0, 1)$ and $u = 1$. The existence proof relies on a regularization of the traveling wave problem allowing us to use well-known existence results [3] for traveling front solutions of continuous neural field equations. Regarding the uniqueness proof, we use the *squeezing* technique of Chen [1] and rely on comparison principles for the LNFE (1) to show that the traveling front solutions which have nonzero wave speed are unique (up to translation).

Result 2: The spectral properties of the traveling fronts are also investigated via a careful study of the linear operator around a traveling front in co-moving frame where we crucially use Fredholm properties of nonlocal differential operators previously obtained in an earlier work [5]. For the spectral analysis, we need to impose an extra exponential localization condition on the interactions.

The main difficulty of this work resides in the infinite range interactions of the connections in (1) where the nonlinearity resides, which prevents us to use classical results from lattice differential equations. There are many interesting developments of the present work: (i) extending the spectral analysis to a nonlinear stability result, (ii) extending the framework to monostable kinetics, (iii) including linear adaptation to study traveling pulses, and (iv) looking at other types of networks such as homogenous trees or Erdös Réyni graphs.

References

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