A singular mean–field interaction with memory. Application to neuroscience

Milica Tomasevic, Inria SAM, milica.tomasevic@inria.fr
Denis Talay, Inria SAM, denis.talay@inria.fr.

Chemotaxis is the collective movement of a population when triggered by presence of a chemical stimulus in their environment. It plays an important role in many biological processes (healing, foraging, immune response). In neuroscience, chemotaxis is crucial for axon growth: it enables the precise guidance of axonal growth cones to their targets [3]. Recently, Calvez and Khonsari [2] proposed a chemotactic approach when modelling a rare type of multiple sclerosis that manifests itself in concentric demyielination patterns.

It has been widely validated that chemotaxis is succesfully modeled on the macroscopic level by the Keller-Segel system. It couples in a non linear way the population density ρ and the concentration of the chemical c:

$$\begin{cases} \partial_t \rho(x,t) = \nabla \cdot (\nabla \rho - \chi \rho \nabla c), \quad t > 0,\ x \in \mathbb{R}^d \\ \partial_t c(x,t) = \triangle c - \lambda c + \rho, \quad t > 0,\ x \in \mathbb{R}^d, \end{cases} \tag{1}$$

Motivated by the study of this fully parabolic model using probabilistic methods, we give rise to a non linear SDE of McKean-Vlasov type with a highly non standard and singular interaction:

$$\begin{cases} dU_t = b(t,U_t)dt + \left\{ \int_0^t (K_{t-s} \star \rho_s)(U_t) ds \right\} dt + dW_t, \quad t \leq T, \\ p_s(y)dy := \mathcal{L}(U_s), \quad X_0 \sim p_0(x)dx, \end{cases} \tag{2}$$

where T is an arbitrary time horizon and $K_t(x) = -\frac{x}{t^{3/2}} e^{-\frac{x^2}{2t}}$. On the particle system level, this interpretation reads

$$\begin{cases} dU_t^{i,N} = b(t, U_t^{i,N})dt + \left\{ \frac{1}{N} \sum_{j=1, j\neq i}^N \int_0^t K_{t-s}(U_t^{i,N} - U_s^{j,N}) ds \right\} dt + dW_t^i, \quad t \leq T \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1 \leq i \leq N), \end{cases} \tag{3}$$

where (W_t^i) are N independent one-dimensional standard Brownian motions. At each time particle i is influenced in a singular way by all the past of all the other particles.
From the probabilistic viewpoint it seems that such interaction has not been studied before in the literature. Adapting the classical techniques of Sznitman [4], one would be able to treat the case of a regular interaction kernel K and prove the propagation of chaos of (3) towards (2). However, additional arguments are required for a singular kernel.

In [5], we prove the existence and uniqueness of a weak solution to (2) in $d = 1$ with the assumption that initial distribution has a density and that b is uniformly bounded. To get this result we use an accurate estimate on densities of one-dimensional diffusions with bounded measurable drifts which is obtained by stochastic methods and allows us to control the L^∞ norms of the time marginal densities of processes constructed in a Picard procedure; thanks to these controls we are able to get uniform bounds on the sequence of drifts, which is essential both to tightness arguments and to get uniqueness of the local solution to the non-linear martingale problem solved by any limit of the Picard procedure. The most important assumption on K is the fact that it belongs to $L^1((0,T]; L^1(\mathbb{R})) \cap L^1((0,T]; L^2(\mathbb{R}))$.

In [1], we prove System (3) is well-posed and it propagates chaos to the unique weak solution of (2) in $d = 1$. The difficulties arising from the singular interaction can be resolved by using purely Brownian techniques. Due to the singular nature of the kernel K, we need to introduce a partial Girsanov transform of the particle system in order to obtain uniform in N bounds for moments of the corresponding exponential martingale. This trick may be useful for other singular interactions.

References

[1] J.F. Jabir, D. Talay and M. Tomašević Mean-field limit of a particle approximation of the one-dimensional parabolic-parabolic Keller-Segel model without smoothing, *Submitted* 2017.

[2] R. Khonsari, V. Calvez, Mathematical description of concentric demyelination in the human brain: self-organization models, from Liesegang rings to chemotaxis., *Math. Comput. Modelling vol. 47, (78), 2008, pp. 726-742.*

[3] D. Mortimer, T. Fothergill, Z. Pujic, L. Richards, G. Goodhill, Growth cone chemotaxis., *Trends in Neurosciences, vol. 31(2), 2008 pp. 90-98.*

[4] A. Sznitman, Topics in propafation of chaos, *École d'Été de Probabilités de Saint-Flour XIX—1989, Lecture Notes in Math., vol. 1464, Springer, Berlin, 1991, pp. 165–251.*

[5] D. Talay and M. Tomašević, A new stochastic interpretation of Keller-Segel equations: the one–dimensional case., *Submitted* 2017.